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Scattering of Poincaré waves by an irregular coastline

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This paper discusses the theory of the reflexion and scattering by an irregular coastline of a Poincaré-type wave on a rotating ocean. It is assumed that the coast is straight except for small deviations from the rectilinear form, and that these deviations may be regarded as a random function of position along the coast. The rigorous theory of energy-transfer processes in random media is applied to determine the power flux from the incident Poincaré wave into the scattered Kelvin wave, which propagates in a unique direction along the coast, and into Poincaré ocean wave noise. The relative efficiencies of generation of these waves is examined in some detail, and studied in particular for varying ranges of values of certain non-dimensional parameters characterizing the coastal configuration. Detailed estimates are given for a shoreline whose irregularities are specified by a Gaussian spectrum of Fourier components, and the results extrapolated in the concluding section of the paper to give a general qualitative discussion of the effects of an arbitrary coastline on an incident wave.

1. Introduction

When a freely propagating *Poincaré* wave (see equations (3.1) and (3.2) below) impinges on a coastline it is subject to several distinct attenuation mechanisms. Of course in the case of an extensive and essentially straight coast most of the incident energy is specularly reflected. However, if the shoreline is irregular a significant amount of energy can be dissipated or scattered. Local small-scale topographic features near the shore are responsible for small-scale turbulent mixing and consequent dissipation of wave energy into heat. For long waves this mechanism is generally unimportant compared with viscous eddy diffusion in the coastal boundary layer. The latter is also enhanced by the presence of larger-scale topographic features in the following sense. Coastal irregularities generate a scattered wave field which may be divided into two distinct parts. The first is the relatively low-frequency 'ocean wave noise', with characteristic periods of hours, and consists of a random field of Poincaré waves scattered out of the incident waves by the coastal irregularities. Second, the same process is responsible for the generation of *Kelvin* waves whose energy is trapped against the coast and thereby dissipated in the coastal boundary layer.

Pinsent (1972) has recently studied the effects of coastal irregularities on incident Poincaré waves. His results are based on a second-order perturbation expansion in powers of a small parameter describing the relative magnitude of the coastal irregularities. Such an approach would not be expected to be appropriate in situations involving extensive coastlines because of the occurrence of secular terms. However, the presence of scattered Poincaré and Kelvin waves is confirmed.

In this paper an attempt is made to generalize the work of Pinsent in order to deal with extensive coastlines. Such problems are conveniently analysed in terms of the theory of wave propagation in random media discussed by Howe (1971, 1972*a*). Actually we shall consider a somewhat simplified version of Pinsent's problem in that it will be assumed that the ocean has constant depth. Inclusion of variable depth would be too difficult in the present approach, but it is argued that, in spite of this, our conclusions regarding the partition of energy between the scattered Kelvin and Poincaré modes would have general indicative value for actual coastlines. We shall also assume that the irregularities of an almost rectilinear coastline may be regarded as a *stationary random function* of position along the coast. For an incident Poincaré wave we shall derive an expression for the reflexion coefficient of the specularly reflected *coherent* wave (§ 3). This result may alternatively be obtained by summing an infinite subseries of secular terms in a formal perturbation expansion (cf. Frisch 1967).

In § 4 an expression is obtained for the power flux into the scattered field per unit length of coast. A remarkable feature of this result is that when attention is confined to the mean flux per unit length of coast, the expression reveals unambiguously, and without recourse to the asymptotic evaluation of Fourier integrals, the manner in which the scattered, random wave energy is distributed amongst the Poincaré and Kelvin wave modes. The mathematical and physical concepts underlying these energy-transfer mechanisms are discussed in some detail.

The results obtained in §§ 3 and 4 are analysed in the geophysical context in § 5. In particular an estimate is obtained for the proportion of the incident energy lost to the scattered field. Also the variation in intensity of the scattered Kelvin waves, as a function of the angle of incidence and frequency of the incident Poincaré wave, is examined for a *Gaussian* coastline. Finally the partition of the scattered energy between the Kelvin and Poincaré modes is discussed.

2. Formulation of the scattering problem

We adopt a system of rectangular co-ordinate axes with respect to which the shallow-water wave equations for a uniformly rotating fluid assume the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - fv + g\frac{\partial \phi}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + fu + g\frac{\partial \phi}{\partial y} &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{h}\frac{\partial \phi}{\partial t} &= 0. \end{aligned}\right\} \qquad (2.1)$$

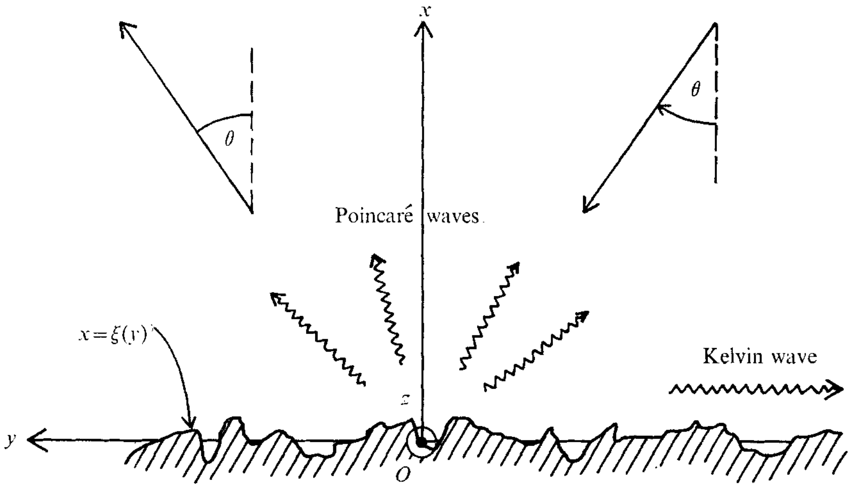

FIGURE 1. Irregular coastal configuration. The whole system rotates about a vertical axis with angular velocity $\frac{1}{2}f$. A Poincaré wave incident at an angle θ to the mean normal to the coastline generates (i) a specularly reflected Poincaré wave, (ii) diffusely scattered Poincaré ocean wave noise, (iii) a coastal Kelvin wave.

where (u, v) is the horizontal fluid velocity and ϕ the surface elevation of the fluid above the undisturbed level; f, g and h respectively denote the Coriolis parameter, the acceleration due to gravity and the undisturbed depth. Without loss of generality it is assumed that $f > 0$.

Taking the Fourier transform of equations (2.1) with respect to time, i.e. assuming a time dependence of the form $e^{-i\omega t}$, we deduce that ϕ satisfies

$$\nabla^2\phi + \frac{\omega^2 - f^2}{s^2}\phi = 0. \tag{2.2}$$

This describes the propagation of disturbances over the water. In this equation $s = (gh)^{\frac{1}{2}}$ is the propagation velocity of long waves in a non-rotating ocean.

It will be assumed that the water occupies a semi-infinite region with $x > 0$ of the x, y plane (see figure 1), and that it is bounded by an irregular coastline specified by

$$x = \xi(y), \tag{2.3}$$

where $\xi(y)$ is in some sense *small*, and also a *stationary random function* of y with *zero mean*, i.e. $\overline{\xi(y)} = 0$, an overbar denoting an average over an ensemble of statistically equivalent coastlines. The boundary condition to be imposed on the coast follows by requiring that the fluid velocity normal to the shore be identically zero, i.e.

$$u = v(\partial\xi/\partial y) \quad \text{on} \quad x = \xi(y). \tag{2.4}$$

We shall be interested in energy-exchange processes associated with the interaction of the shore with an incident Poincaré wave, so that in approximating (2.4) by an equivalent condition imposed on $x = 0$ rather than $x = \xi(y)$ care must be taken to include all terms likely to be of importance in these processes.

Since ξ is small we expand the first equation of (2.4) in powers of ξ and stop after terms involving ξ^2:

$$u = v\xi_y - u_x\xi - \tfrac{1}{2}u_{xx}\xi^2 + v_x\xi\xi_y \quad \text{on} \quad x = 0. \tag{2.5}$$

Now the fundamental effect of a randomly varying coastline on an incident wave is the generation of a whole spectrum of wavelike disturbances which are randomly distributed in direction. Physically it is anticipated that this scattered distribution of waves will separate into two parts, the first being a random distribution of Poincaré waves, which radiate a portion of the incident energy into all directions away from the coastline, and the second consisting of a system of Kelvin waves which propagate along the coast in one direction. We shall argue (§ 4) that in practice the energy of these Kelvin waves is dissipated in the coastal boundary layer before multiple scattering becomes an important issue.

It would seem to be desirable, therefore, to adopt a representation of the fluid motions which distinguishes between *mean* or *coherent* motions and those associated with the randomly scattered wave field. To do this we introduce an ensemble average wave field, denoted by $\overline{\phi}$, which, as above, represents a mean in the sense of an average taken over an ensemble of statistically equivalent coastlines. In a particular realization of the wave field a correction must be applied to this *coherent* field in order to describe precisely the actual state of the fluid motions. This correction, denoted by ϕ', is the *random* component of the field, and

$$\phi = \overline{\phi} + \phi'. \tag{2.6}$$

Similar partitions apply to the velocity components u and v.

By taking the ensemble average of the wave equation (2.2) we deduce that

$$\left.\begin{aligned} \nabla^2\overline{\phi} + \left(\frac{\omega^2 - f^2}{s^2}\right)\overline{\phi} &= 0, \\ \nabla^2\phi' + \left(\frac{\omega^2 - f^2}{s^2}\right)\phi' &= 0. \end{aligned}\right\} \tag{2.7a, b}$$

As expected from the principle of superposition, the mean and random wave fields separately satisfy the wave equation. The coupling between the two fields is effected by the random boundary condition (2.5); this condition contains the physics of the scattering processes.

Now from (2.1) we can deduce the following well-known expressions for the velocity field:

$$\left.\begin{aligned} u &= g[f\phi_y - i\omega\phi_x]/(\omega^2 - f^2), \\ v &= -g[f\phi_x + i\omega\phi_y]/(\omega^2 - f^2), \end{aligned}\right\} \tag{2.8}$$

and these may be used to express the boundary condition (2.5) in terms of ξ and ϕ alone. Suppose that this condition is represented formally by

$$\mathscr{L}\phi = G_1\phi + G_2\phi, \tag{2.9}$$

where $\mathscr{L}$ and G_1 and G_2 are *linear operators*. The operator $\mathscr{L}$ is assumed to be *non-random*, and $\mathscr{L}\phi = 0$ would therefore be the boundary condition appropriate to a perfectly straight coast $x = 0$; the operator G_1 involves the random function ξ

linearly, and G_2 contains terms quadratic in ξ. Clearly $\overline{G}_1 = 0$, but $\overline{G}_2 \neq 0$, since $\overline{\xi^2} \neq 0$, and we may therefore set $G_2 = \overline{G}_2 + G_2'$.

Now take the ensemble average of (2.9):

$$\mathscr{L}\overline{\phi} = \overline{G_1\phi'} + \overline{G}_2\overline{\phi} + \overline{G_2'\phi'}. \tag{2.10}$$

Subtracting this from (2.9) gives

$$\mathscr{L}\phi' = G_1\overline{\phi} + G_2'\overline{\phi} + \{G_1\phi' - \overline{G_1\phi'}\} + \{G_2\phi' - \overline{G_2\phi'}\}. \tag{2.11}$$

If the mean field $\overline{\phi}$ is assumed to be known then condition (2.11) may be regarded as a boundary condition describing the *generation* of the random field by the interaction of the mean field with the irregularities of the coast. Actually the bracketed terms in (2.11) themselves describe interaction processes of the scattered field and these irregularities, i.e. multiple scattering effects. These terms are important in as much that it would *not* be permissible to neglect them in any calculation of the random field *per se*. However, when attention is focused on the properties of the mean field it is observed that the boundary condition (2.10) for $\overline{\phi}$ involves *correlations* such as $\overline{G_1\phi'}$ of the irregularities and the scattered waves. It is clear that the only non-trivial contributions come from those constituent waves of ϕ' which have been scattered within a *correlation length* L, say, of the point on the coast at which $\overline{G_1\phi'}$ is to be evaluated. Here the length L refers to the correlation scale of the random irregularities $\xi(y)$. Provided only that multiple scattering is not important over distances of order L, and this can always be guaranteed for sufficiently small ξ, it is therefore valid to neglect the bracketed terms on the right-hand side of (2.11) *when the scattered field* ϕ' *is to be used to evaluate the correlation products of* (2.10). Thus for this purpose we may write

$$\mathscr{L}\phi' = G_1\overline{\phi} + G_2'\overline{\phi} \tag{2.12}$$

on $x = 0$.

When this condition is used to determine ϕ' it is seen that the result has a contribution of $O(\xi)$ from the first term on the right, and one of $O(\xi^2)$ from the second. This solution when substituted into the right-hand side of (2.10) will therefore give terms of $O(\xi^2)$ and higher. For sufficiently small ξ the higher-order terms may be neglected in comparison with the lowest-order correction to the rectilinear coastline. This means that (2.10) and (2.12) may be further approximated by

$$\mathscr{L}\overline{\phi} = \overline{G_1\phi'} + \overline{G}_2\overline{\phi}, \quad \mathscr{L}\phi' = G_1\overline{\phi}. \tag{2.13}$$

For an alternative discussion of these points see Howe (1971).

If we now insert explicit expressions for the operators appearing in (2.13) we obtain respectively

$$f\overline{\phi}_y - i\omega\overline{\phi}_x = -\overline{[f\phi_x' + i\omega\phi_y']\xi_y} - \overline{[f\phi_{xy}' - i\omega\phi_{xx}']\xi} - \tfrac{1}{2}[f\overline{\phi}_{xxy} - i\omega\overline{\phi}_{xxx}]\overline{\xi^2} \tag{2.14}$$

and
$$f\phi_y' - i\omega\phi_x' = -[f\overline{\phi}_x + i\omega\overline{\phi}_y]\xi_y - [f\overline{\phi}_{xy} - i\omega\overline{\phi}_{xx}]\xi \tag{2.15}$$

on $x = 0$.

The procedure to be adopted in using these boundary conditions is as follows. Assume that $\overline{\phi}$ is known. Use (2.15) in conjunction with the wave equation (2.7*b*) to determine ϕ' in terms of $\overline{\phi}$. When this solution is substituted into (2.14) we obtain a boundary condition involving the mean wave $\bar{\phi}$ alone. This analysis will be carried through in the next section for the case of an incident Poincaré wave interacting with the coastline.

3. The specularly reflected field

Consider now a plane Poincaré wave specified by

$$\bar{\phi}_I = \exp\left[i(-l_0 x + m_0 y - \omega t)\right] \tag{3.1}$$

incident on the coastline in the manner illustrated in figure 1. In this definition l_0 and ω are assumed to be positive and l_0, m_0 and ω satisfy the *dispersion relation*

$$l_0^2 + m_0^2 = (\omega^2 - f^2)/s^2 \quad (>0) \tag{3.2}$$

obtained by formal substitution of (3.1) into the wave equation (2.2). In taking an average over an ensemble of irregular coastlines the field $\bar{\phi}_I$ is *statistically invariant*, i.e. $\bar{\phi}_I$ is part of the *coherent* wave field. Denote the coherent reflected field by

$$\bar{\phi}_R = R \exp\left[i(l_0 x + m_0 y - \omega t)\right], \tag{3.3}$$

where R is an appropriate reflexion coefficient. That this is an admissible form for the coherent reflected wave will be clear from the ensuing analysis.

Thus we may now set (dropping explicit mention of the time dependence)

$$\bar{\phi} = \exp\left[i(-l_0 x + m_0 y)\right] + R \exp\left[i(l_0 x + m_0 y)\right]. \tag{3.4}$$

When this is substituted into the right-hand side of the boundary condition (2.15) governing the generation of the random field we obtain

$$\left.\begin{aligned} f\phi'_y - i\omega\phi'_x &= (C\xi_y - B\xi)\exp\left[im_0 y\right], \\ \text{where} \qquad C &= \omega m_0(1+R) + il_0(1-R), \\ B &= fm_0 l_0(1-R) + i\omega l_0^2(1+R). \end{aligned}\right\} \tag{3.5}$$

However, in $x > 0$ the random field ϕ' satisfies the wave equation (2.7*b*), and the appropriate solution satisfying the radiation condition may be expressed formally as a Fourier integral over wavenumbers m conjugate to y, namely

$$\phi' = \int_{-\infty}^{\infty} A(m)\exp\left[i(lx + my)\right]dm, \tag{3.6}$$

say, where

$$l = \left(\frac{\omega^2 - f^2}{s^2} - m^2\right)^{\frac{1}{2}}, \tag{3.7}$$

and l is *positive* when real and *positive imaginary* otherwise.

If the Fourier transform with respect to y of the boundary condition (3.5) is now taken and the solution (3.6) is then used to determine $A(m)$, it is an easy matter to deduce that

$$\phi' = \frac{1}{2\pi}\int\!\!\int_{-\infty}^{\infty} \frac{[i(m-m_0)C - B]\,\xi(X)}{\omega l + imf}\exp\left[i(lx+my) - iY(m-m_0)\right]dm\,dY. \tag{3.8}$$

Note again that this formal solution is only suitable for determining correlation products in the manner to be described below. In (3.8) the pole at the zero of

$\omega l + imf$, namely at $m = -\omega/s$ is avoided by indenting the contour of integration in the m plane to pass *above* the singularity. This ensures that the radiation condition is satisfied. Equation (3.8) expresses the random field ϕ' as a summation over all possible plane wave modes satisfying (3.7).

Now the irregularities $\xi(y)$ have been assumed to constitute a *stationary* process, so that there exists a correlation function $\mathscr{R}$, say, given by

$$\mathscr{R}(y-Y) = \overline{\xi(y)\,\xi(Y)}, \tag{3.9}$$

which is an *even* function of $y-Y$. The *spectrum* of the random function ξ is a non-negative function defined by

$$\Phi(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{R}(y)\exp[-i\kappa y]\,dy, \tag{3.10}$$

the Fourier transform of the correlation function.

Hence if the formal solution ϕ' given by (3.8) is substituted into the mean-wave boundary condition (2.14), with $\overline{\phi}$ replaced by (3.4), and the result divided through by $\exp[im_0 y]$, we have

$$(1-\tfrac{1}{2}l_0^2\overline{\xi^2})\{im_0 f(1+R)+\omega l_0(1-R)\}$$
$$= \int_{-\infty}^{\infty} \frac{\{i(m-m_0)\,C-B\}\{m_0(lf+i\omega m)-i\omega(\omega^2-f^2)\,s^{-2}\}}{\omega l+imf}\,\Phi(m-m_0)\,dm. \tag{3.11}$$

Substituting for C, B from (3.5), and solving the resulting equation for the reflexion coefficient R, we finally deduce that

$$R = \frac{(1-\tfrac{1}{2}l_0^2\overline{\xi^2})(\omega l_0 - im_0 f) - (l_0 f - i\omega m_0)\,\sigma_1 - i\omega(\omega^2-f^2)\,s^{-2}\,\sigma_2}{(1-\tfrac{1}{2}l_0^2\overline{\xi^2})(\omega l_0 + im_0 f) - (l_0 f + i\omega m_0)\,\sigma_1 + i\omega(\omega^2-f^2)\,s^{-2}\,\sigma_2}, \tag{3.12}$$

where σ_1 and σ_2 are $O(\xi^2)$ complex-valued functions defined by

$$\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \int_{-\infty}^{\infty}\begin{pmatrix}m\\ 1\end{pmatrix}\frac{[m_0(lf+i\omega m)-i\omega(\omega^2-f^2)\,s^{-2}]\,\Phi(m-m_0)}{\omega l+imf}\,dm. \tag{3.13}$$

Thus (3.12) is the desired expression for the reflexion coefficient of the specularly reflected coherent wave. We shall see later that $|R| < 1$, showing that the energy reflected in the coherent mode is less than that of the incident wave. The balance, of course, is made up by taking account of the energy scattered into the random wave modes. The analysis of this energy-transfer process constitutes the substance of § 4. Note, however, that when ξ is set equal to zero in (3.12) we recover the following well-known expression for the reflexion coefficient for reflexion by a rectilinear coastline:

$$R = \frac{\omega l_0 - im_0 f}{\omega l_0 + im_0 f}. \tag{3.14}$$

In this case $|R| = 1$, as expected.

4. Energy exchanges in scattering

In the previous section the reflexion coefficient for an incident plane Poincaré wave was calculated. That calculation is expected to be rather efficient for determining the properties of the mean field. However, it has been pointed out that a complete discussion of the random component of the wave field, ϕ', requires the use of the full boundary condition, represented formally by (2.11), which includes effects of multiple scattering of the random waves by the irregular coastline. Work is currently in hand on this aspect of the problem, and is based on the so-called *kinetic theory* of wave propagation in random media (Howe 1972*b*). Here we shall confine ourselves to a rigorous discussion of the energy exchange processes between the mean and random fields. In particular we shall determine respectively expressions for the energy flux into scattered Poincaré and Kelvin waves. These Poincaré waves are unlikely to be significantly affected by multiple scattering because they tend, on the whole, to be radiated directly away from the coast. A Kelvin wave, however, is continuously subjected to further scattering. We shall argue that these multiple scattering effects are generally unimportant inasmuch as they affect the attenuation of the Kelvin wave modes. Actually such scattering is expected to proceed over distances and times of order $1/|\overline{\xi^2}|$ (cf. Pinsent 1972). These scales are large compared with the incident wavelength and period, and we shall take the view that they are also large in comparison with the corresponding scales associated with dissipation in the coastal boundary layer.

To analyse the energy exchange processes we return to the quadratic boundary condition (2.5). In accordance with the discussion of the boundary conditions given in § 2, the quadratic terms in ξ may be replaced by their ensemble averages:

$$u = v\xi_y - u_x\xi - \tfrac{1}{2}u_{xx}\overline{\xi^2} + v_x\overline{\xi\xi_y},$$

i.e.

$$u = v\xi_y - u_x\xi - \tfrac{1}{2}u_{xx}\overline{\xi^2}, \tag{4.1}$$

since

$$\overline{\xi\xi_y} \equiv (\partial\mathscr{R}(y)/\partial y)_{y=0} \equiv 0.$$

In the absence of irregularities in the coastline the right-hand side of (4.1) vanishes identically. The consequent vanishing of u on $x = 0$ then implies that no energy is lost from the 'coherent' field.

Take the ensemble average of (4.1):

$$\overline{u} = \overline{v'\xi_y} - \overline{u'_x\xi} - \tfrac{1}{2}\overline{u}_{xx}\overline{\xi^2}, \tag{4.2}$$

and subtract this from (4.1):

$$u' = \overline{v}\xi_y + \{v'\xi_y - \overline{v'\xi_y}\} - \overline{u}_x\xi - \{u'_x\xi - \overline{u'_x\xi}\} - \tfrac{1}{2}u'_{xx}\overline{\xi^2}. \tag{4.3}$$

Now let $p = \rho g\phi$ denote the pressure perturbation due to the wave motions. Then if (4.2) is multiplied by $-\overline{p}$ (at $x = 0$), giving

$$-\overline{p}.\overline{u} = -\overline{p}(\overline{v'\xi_y} - \overline{u'_x\xi}) + \tfrac{1}{2}\overline{p}.\overline{u}_{xx}\overline{\xi^2}, \tag{4.4}$$

the following interpretation is in order. The left-hand side represents the rate at which the mean wave field is doing work on the effective coastline at $x = 0$. This would be identically zero if the coast were perfectly straight, but the presence of irregularities represented by the right of (4.4) gives a non-zero result. This corresponds to the rate at which energy is supplied to the wave motions of the random field.

Denote this power flux by $-P_M$, where

$$P_M = \bar{p}(\overline{v'\xi_y} - \overline{u'_x\xi}) - \tfrac{1}{2}\bar{p}\,.\,\bar{u}_{xx}\overline{\xi^2}. \tag{4.5}$$

Similarly if (4.3) is multiplied by $p' = \rho g\phi'$ and the ensemble average is taken we deduce that

$$\overline{p'u'} = \bar{v}\overline{p'\xi_y} - \bar{u}_x\overline{p'\xi} + \{\overline{p'v'\xi_y} - \overline{p'u'_x\xi} - \tfrac{1}{2}\overline{p'u'_{xx}\xi^2}\}. \tag{4.6}$$

In this expression $\overline{p'u'}$ is equal to the rate at which the effective boundary at $x = 0$ does work on the random wave field. Actually the terms in braces on the right-hand side of (4.6) involve multiple scattering of random waves, and account for the *redistribution* of random energy amongst the available modes. The first two terms, on the other hand, represent the rate at which energy is supplied to the random field owing to the interaction of the mean field with the coastline. This will be denoted by P_R:

$$P_R = \bar{v}\overline{p'\xi_y} - \bar{u}_x\overline{p'\xi}. \tag{4.7}$$

Naturally we hope to verify that in some sense $P_M + P_R = 0$, corresponding to energy conservation in the exchange process. This result is indeed true when the functions P_M and P_R are averaged over a wavelength and period of the incident coherent wave, i.e. denoting such averages by $\langle\,\rangle$ we shall see that

$$\langle P_M\rangle + \langle P_R\rangle = 0. \tag{4.8}$$

To carry through this procedure we again consider the case of an incident Poincaré wave of the form given in (3.4). Actually we must be careful to deal with real-valued quantities when undertaking energy calculations, and we therefore modify $\bar{\phi}$ by the addition of a complex conjugate (c.c.) and adjustment of the amplitude:

$$\bar{\phi} = \tfrac{1}{2}\{\exp[-il_0x] + R\exp[il_0x]\}\exp[im_0y] + \text{c.c.} \tag{4.9}$$

The incident wave is now real and of unit amplitude.

Let us first use (4.7) to determine $\langle P_R\rangle$. Now arguments used in §2 may be applied to show that the ensemble averages appearing in (4.7) may be evaluated by means of the *local Born approximation* solution given by equation (3.8). The random pressure fluctuation p' is equal to $\rho g\phi'$. The mean field velocity terms $\bar{v}$ and $\bar{u}_x$ appearing in (4.7) are expressed in terms of $\bar{\phi}$ using the mean of equations (2.8). Carrying through the analysis, with the reflexion coefficient given by (3.12), we eventually deduce that, correct to $O(\xi^2)$,

$$\langle P_R\rangle = \frac{\rho g^2 l_0^2}{\omega^2/s^2 - m_0^2}\int_{-\infty}^{\infty}\Phi(m-m_0)\left[mm_0 - \frac{\omega^2}{s^2}\right]^2\left\{\frac{1}{\omega l + imf} + \frac{1}{(\omega l + imf)^*}\right\}dm, \tag{4.10}$$

where the asterisk denotes the complex conjugate.

Now this integral has a very interesting structure, since the terms in curly brackets show that only the *real* part of the function $\omega l + imf$ is important in the energy-transfer process. When this expression is examined it is seen that the only non-trivial contributions from that bracket are from two sources:

(i) the pole at $\omega l + imf = 0$,

(ii) the interval $|m| < [(\omega^2 - f^2)/s^2]^{\frac{1}{2}}$ over which $l = \{[(\omega^2 - f^2)/s^2] - m^2\}^{\frac{1}{2}}$ is real.

Recalling the physical interpretation of the integral in equation (3.8) as a summation over all possible random wave motions scattered by the interaction between the mean wave field and the irregularities of the coastline, we conclude the *the only random wave modes contributing to the energy transfer process are those labelled* (i) *and* (ii) *above.*

From (i), and using expression (3.7) for l, we have $m = -\omega/s$, and this term therefore gives the energy scattered into a wave mode which is *trapped* against the shoreline (since l is purely imaginary), i.e. the energy scattered into the *Kelvin wave mode*:

$$\langle P_R \rangle_K = \frac{2\pi f \rho g \omega^2 l_0^2}{h(\omega^2 - f^2)} \frac{(\omega/s + m_0)^2}{\omega^2/s^2 - m_0^2} \Phi\left(\frac{\omega}{s} + m_0\right). \tag{4.11}$$

Note that this is positive definite, and vanishes if $f = 0$, i.e. in the absence of rotation.

Similarly, the waves associated with (ii) consist of all *propagating* Poincaré modes. Using (4.10) it follows that their contribution is precisely

$$\langle P_R \rangle_P = \frac{2\rho g^2 l_0^2 \omega}{(\omega^2/s^2 - m_0^2)(\omega^2 - f^2)} \int_{-\kappa_0}^{\kappa_0} \frac{\Phi(m - m_0)(m m_0 - \omega^2/s^2)^2 |\kappa_0^2 - m^2|^{\frac{1}{2}}}{\omega^2/s^2 - m^2} dm, \tag{4.12}$$

where $\kappa_0 = |(\omega^2 - f^2)/s^2|^{\frac{1}{2}}$. Again this result is positive definite. Thus $\langle P_R \rangle$ reduces to the sum of the two terms given by (4.11) and (4.12).

The remarkable feature of these results is that they have been derived independently of arguments familiar in classical scattering theory. That theory determines the energy scattered into the various permissible wave modes by first evaluating the solution (3.8) for ϕ' asymptotically for large x or y. The main contribution then comes from the pole (i), and from stationary-phase terms involving waves in the region (ii). The present approach demonstrates unequivocally that, provided we are interested in ensemble average energy fluxes, such an asymptotic analysis is redundant, since the only possible wave modes capable of supporting propagating energy *must* be those given by (i) and (ii). This is the physical significance of the term in curly brackets in (4.10). For a discussion along these lines of scattering processes in more general random media reference may be made to Howe (1972*a*).

A similar analysis can be applied to the right-hand side of (4.5) to determine $\langle P_M \rangle$. The last term on the right of (4.5) turns out to be of $O((\overline{\xi^2})^2)$ and is therefore neglected, but the first two terms reduce precisely to the result (4.10) with the sign changed. This confirms our earlier statement that $\langle P_M \rangle + \langle P_R \rangle = 0$. Since $\langle P_M \rangle$ is negative definite the energy exchanges are unidirectional, from the mean field to the random field, and *not* vice versa. This is in accordance with intuitive notions and the *Second Law of Thermodynamics*.

Note also that $\langle P_M \rangle$ may alternatively be determined directly from the properties of the mean field obtained in the previous section. Indeed from (4.4) we have

$$\langle P_M \rangle = \langle \overline{p} . \overline{u} \rangle, \tag{4.13}$$

where

$$\overline{p} = \tfrac{1}{2}\rho g(1 + R)\, e^{im_0 y} + \text{c.c.},$$

and $$\bar{u} = \frac{g}{2(\omega^2-f^2)}\{R(\omega l_0+im_0f)-(\omega l_0+im_0f)\}\exp[im_0y]+\text{c.c.}$$

at $x = 0$. Hence it follows that

$$\langle P_M\rangle \equiv \langle \bar{p}.\bar{u}\rangle = \frac{-\rho\omega l_0 g^2}{2(\omega^2-f^2)}(1-|R|^2). \tag{4.14}$$

This result is of the form familiar in the calculation of energy loss rates in reflexion processes. Using the formula derived in § 3 for the reflexion coefficient R (equation (3.12)) together with the integral formulae (3.13), it is a simple, yet tedious, matter to show that (4.14) leads to an expression for $\langle P_M\rangle$ identical to that obtained above. However this approach lacks the precision of interpretation of that based on the direct calculation of the interaction terms given by (4.5) and (4.7). This is because in computing the latter, the random wave field enters directly in the form of an integral over scattered waves representing p', say, and the corresponding expression (4.10) is readily seen to describe unambiguously the manner in which these scattered modes contribute to the energy-transfer process. On the other hand the evaluation of (4.14) involves the use of the integral formulae (3.13) which are, to be sure, associated with the scattering mechanism, but enter (4.14) essentially by two distinct routes. The first of these is in the expression for $\bar{p}$ and the second in that for $\bar{u}$. Thus, although the formal conclusion is identical with that obtained by the direct method, the procedure fails to illuminate the underlying interaction mechanisms, expressed by the right-hand sides of (4.5) and (4.7), responsible for the channelling of the scattered energy into the propagating Poincaré and Kelvin modes.

5. Geophysical implications of the theory

In §§ 3 and 4 expressions have been derived for the specular reflexion coefficient R and the rates $\langle P_R\rangle_K$ and $\langle P_R\rangle_P$ at which energy is transferred from an incident Poincaré wave to the scattered Kelvin and Poincaré wave fields. We now proceed to apply these results to an analysis of certain geophysically significant questions.

For a given irregular coastline it is desirable to be able to estimate the portion of the energy content of an incident wave system that is scattered into the Poincaré 'ocean wave noise' and into the Kelvin wave modes. This is effectively measured by the magnitude of $1-|R|^2$. Also, while it is clear that Poincaré waves are always generated by coastal irregularities, it is not always the case that a significant Kelvin wave field will be developed. Actually the efficiency of Kelvin wave generation is critically dependent on the form of the spectrum function $\Phi(m)$ of the coast. We shall see that a Kelvin wave is only generated when a certain 'resonant interaction condition' can be satisfied by the incident wave and the Fourier components of the coastal irregularities. In this context it is natural to enquire into the variation of the energy flux into the Kelvin modes as the frequency and direction of propagation of the incident wave change. This is important because most of the Kelvin wave energy is subsequently dissipated in the coastal boundary layer, and also because Kelvin waves have often been observed to have surprisingly large amplitudes along certain coasts (see, e.g.

Crease 1956; Munk, Snodgrass & Wimbush 1970). We shall also examine the energy partition ratio $\Theta = \langle P_R \rangle_K / \langle P_R \rangle_P$ giving the ratio of the fluxes into the Kelvin and Poincaré modes. This will be discussed in a semi-analytical manner as a function of the frequency of the incident wave. It should be noted, however, that small values of Θ do not necessarily imply a relatively negligible Kelvin wave field, since the energy of that wave is perforce confined to a fairly narrow coastal channel in which it would be quite possible for Kelvin waves of fairly substantial amplitude to be maintained by a relatively inefficient energy conversion mechanism.

In order to illustrate the role of the spectrum $\Phi(m)$ of the coastal irregularities we shall introduce a cut-off wavenumber $\mu > 0$ such that for $|m| > \mu$ the spectral content is minimal, i.e.

$$\Phi(m) \simeq 0 \quad \text{for} \quad |m| > \mu.$$

This cut-off wavenumber is related to the correlation length L of $\xi(y)$, discussed earlier, by the relation $\mu L \simeq 1$.

First consider equation (4.11) giving the power flux $\langle P_R \rangle_K$ from an incident Poincaré wave into the Kelvin wave mode. From what has been said above it is clear that this flux is significant provided that $|(\omega/s) + m_0| < \mu$, in other words, provided that there exists a significant spectral component κ $(-\mu < \kappa < \mu)$ such that

$$m_0 + \kappa = -\omega/s. \tag{5.1}$$

In this equation m_0 is the coastal component of the wavenumber of the incident Poincaré wave and $-\omega/s$ that of the scattered Kelvin wave. The wavenumber κ is associated with one of the Fourier components of the coast. Hence (5.1) constitutes a *resonant interaction condition* specifying conservation of wavenumber in the coastal direction during wave–wave interactions between the incident wave and the *stationary wave* Fourier components of the coastline. This is the condition which two interacting waves must satisfy in order to achieve a non-zero transfer of energy between propagating modes (see, e.g. Phillips 1960). Note that the related condition requiring conservation of frequencies is automatically satisfied, since the Fourier components of $\xi(y)$ are time-independent.

Consider next the magnitude of the reflexion coefficient R (equation (3.12)). This is readily obtained with a minimum of further analysis by noting that, by equation (4.14),

$$\frac{-\rho\omega l_0 g^2}{2(\omega^2 - f^2)}(1 - |R|^2) = \langle \overline{p} . \overline{u} \rangle$$
$$= -\langle P_R \rangle_K - \langle P_R \rangle_P,$$

i.e.

$$|R|^2 = 1 - E_K - E_P, \tag{5.2}$$

where

$$E_K = \frac{4\pi f \omega l_0 (\omega/s + m_0)}{s^2(\omega/s - m_0)} \Phi\left(\frac{\omega}{s} + m_0\right), \tag{5.3}$$

and

$$E_P = \frac{4 l_0}{(\omega^2/s^2) - m_0^2} \int_{-\kappa_0}^{\kappa_0} \frac{\Phi(m - m_0)(m m_0 - \omega^2/s^2)^2 |\kappa_0^2 - m^2|^{\frac{1}{2}}}{(\omega^2/s^2) - m^2} \, dm. \tag{5.4}$$

Both of these quantities E_K and E_P are positive definite. They are also of $O(\epsilon^2)$, where ϵ is a non-dimensional parameter characterizing the magnitude of the coastal irregularities and which has been assumed to be small in the earlier analysis. Before identifying ϵ and estimating its value in the geophysical context, let us observe that E_K and E_P are respectively equal to the Kelvin and Poincaré wave *efficiency factors* defined by

$$E_K = \frac{\text{energy flux into Kelvin wave}}{\text{incident energy flux}},$$

$$E_P = \frac{\text{energy flux into Poincaré waves}}{\text{incident energy flux}}.$$

The consistency of these definitions follows by noting that the incident energy flux is precisely equal to $\rho\omega l_0 g^2/[2(\omega^2-f^2)]$, and by referring to the formulae (4.11) and (4.12).

The energy partition ratio Θ may also be expressed in terms of these quantities:

$$\Theta = \frac{E_K}{E_P} = \frac{(\pi f\omega/s^2)\,(\omega/s+m_0)^2\,\Phi(\omega/s+m_0)}{\int_{-\kappa_0}^{\kappa_0}\Phi(m-m_0)\,(mm_0-\omega^2/s^2)^2\,|\kappa_0^2-m^2|^{\frac{1}{2}}/(\omega^2/s^2)-m^2\,dm}. \tag{5.5}$$

Let us now express E_K, E_P and Θ in terms of a fundamental set of non-dimensional parameters which are appropriate to specify the coastal configuration. Thus we shall write the spectrum function $\Phi(m)$ in the form

$$\Phi(m) = (\overline{\xi^2}/\mu)\,S(m/\mu), \tag{5.6}$$

where $\overline{\xi^2} = R(0)$ and $S(m/\mu)$ is a dimensionless function of order unity with the properties

$$\left.\begin{aligned} S(m/\mu) &\simeq 0 \quad \text{for} \quad |m/\mu| > 1,\\ \int_{-\infty}^{\infty} S(\lambda)\,d\lambda &= 1. \end{aligned}\right\} \tag{5.7}$$

Introducing a polar representation for the incident Poincaré wave specified by (l_0, m_0), namely

$$(l_0, m_0) = \kappa_0(\cos\theta, \sin\theta), \quad |\theta| < \tfrac{1}{2}\pi, \tag{5.8}$$

reduces equation (3.2), the Poincaré wave dispersion relation, to

$$\kappa_0 = (f/s)\,(\sigma^2-1)^{\frac{1}{2}}, \tag{5.9}$$

where $\sigma = \omega/f > 1$. The formulae (5.3), (5.4) and (5.5) may then be expressed in the following forms:

$$E_K = 4\pi\alpha\beta^2\gamma\sigma\cos\theta\,\frac{\sigma+\gamma\sin\theta}{\sigma-\gamma\sin\theta}\,S(\alpha[\sigma+\gamma\sin\theta]), \tag{5.10}$$

$$E_P = \frac{4\beta^2\gamma\cos\theta}{\alpha(\sigma^2-\gamma^2\sin^2\theta)}\,I, \tag{5.11}$$

$$\Theta = \pi\alpha^2\sigma(\sigma+\gamma\sin\theta)^2\,S(\alpha[\sigma+\gamma\sin\theta)]/I. \tag{5.12}$$

In these expressions the dimensionless parameters α, β, γ and I are defined by

$$\alpha = f/s\mu, \quad \beta = (f/s)\,(\overline{\xi^2})^{\frac{1}{2}}, \quad \gamma = (\sigma^2-1)^{\frac{1}{2}},$$

$$I = \int_{-\alpha\gamma}^{\alpha\gamma} \frac{S(\lambda-\alpha\gamma\sin\theta)\,[\alpha\sigma^2-\lambda\gamma\sin\theta]^2\,(\alpha^2\gamma^2-\lambda^2)^{\frac{1}{2}}\,d\lambda}{\alpha^2\sigma^2-\lambda^2}. \tag{5.13}$$

Now Poincaré and Kelvin waves generally have frequencies ω that are characteristic of storm surges and the M_2 (semi-diurnal) and the K_1 (diurnal) tidal constituents, so that $\sigma = \omega/f \gtrsim O(1)$. From (5.9) we also note that the limiting cases of $\sigma \gg 1$ and $\sigma \to +1$ correspond to very short and very long incident wavelengths respectively. Further we have, typically, $f \simeq 10^{-4}\,\mathrm{s}^{-1}$, $g \simeq 10^3\,\mathrm{cm\,s}^{-2}$, $h \simeq 10^5\,\mathrm{cm}$ and, since most continental coastlines have an extent of at most several thousand kilometres, $1/\mu \simeq L < 10^8\,\mathrm{cm}$, and therefore

$$\alpha = f/\mu(gh)^{\frac{1}{2}} \lesssim O(1).$$

Finally, note that $\beta \lesssim 0{\cdot}1$, since the root-mean-square amplitude of the coastal irregularity, $(\overline{\xi^2})^{\frac{1}{2}}$, is likely to be at most of order 10^2 kilometres.

From (5.10) and (5.11) it is seen that E_K and E_P are both of $O(\beta^2)$, and therefore from (5.2) that $1-|R|^2 = O(\beta^2) \ll 1$. We may therefore identify β with the parameter ϵ referred to above. The smallness of this parameter indicates that in practice a relatively small proportion of the incident wave energy is actually transferred to the scattered field. Further, since E_K, $E_P \propto \gamma = (\sigma^2-1)^{\frac{1}{2}}$, the energy content of the random field is likely to be especially small for frequencies in the inertial range ($\sigma \to +1$), i.e. for very long waves. This is in qualitative agreement with recent tidal observations off the coast of California. Munk *et al.* (1970) observed that along the full extent of the Californian coast the magnitude of the M_2 constituent ($\sigma \simeq 2$) is considerably greater than that of the K_1 constituent ($\sigma \simeq 1$).

In order to examine the significance of our results in a more specific manner we consider a definite form for the spectrum function of the coastal irregularities, viz.,

$$S(m/\mu) = \pi^{-\frac{1}{2}}\, e^{-m^2/\mu^2}. \tag{5.14}$$

Particular interest is attached to the generation efficiency of the Kelvin waves, E_K, which may now be given the explicit form

$$\frac{E_K}{4\pi^{\frac{1}{2}}\beta^2} = \alpha\gamma\sigma\cos\theta\left(\frac{\sigma+\gamma\sin\theta}{\sigma-\gamma\sin\theta}\right)\exp\left[-\alpha^2(\sigma+\gamma\sin\theta)^2\right]. \tag{5.15}$$

It is clear from this expression that when α is large (very long coastal correlation lengths) $E_K \simeq 0$ since $\sigma+\gamma\sin\theta \neq 0$ for $|\theta| < \frac{1}{2}\pi$. In other words, Kelvin waves will not tend to be generated by relatively *smooth* coastlines, in agreement with intuition. In the opposite limit of $\alpha \to 0$ (short correlation lengths), E_K is also small except possibly for very short incident waves (σ large).

In figure 2 the dependence of E_K specified by (5.15) on the angle of incidence θ of the incident Poincaré wave is illustrated for $0{\cdot}2 \leqslant \alpha \leqslant 1{\cdot}0$, and $1{\cdot}1 \leqslant \sigma \leqslant 3{\cdot}0$. Note that each curve has a single maximum, and that in general the curves become more peaked as the frequency σ increases. Also it is apparent that $E_K/4\pi^{\frac{1}{2}}\beta^2 \lesssim O(1)$, and hence E_K itself is indeed small since $\beta^2 \ll 1$. For a fixed value of $\alpha \simeq 1$ (figures 2(a, b)) it will be noted that with increasing σ efficient Kelvin wave generation is associated increasingly with 'forward scatter', in the sense that the peak efficiencies tend to occur for incident waves whose velocities of propagation are in the same direction as that of the Kelvin modes. On the other hand, for smaller values of the correlation scale α, figures 2(d, e) indicate

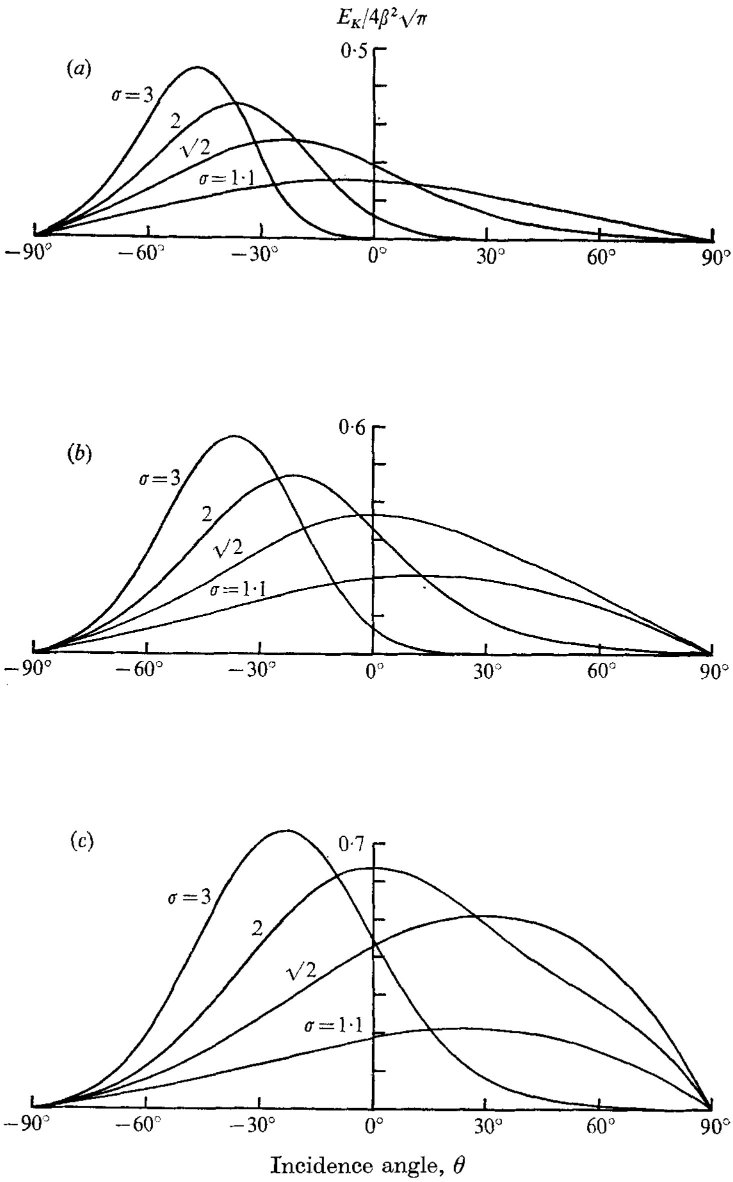

FIGURE 2. The Kelvin wave efficiency factor $E_K/4\beta^2\sqrt{\pi}$, for a coast whose irregularities are characterized by a Gaussian spectrum function. The correlation length non-dimensionalized by $(gh)^{\frac{1}{2}}/f$ is denoted by α, and σ is the incident wave frequency non-dimensionalized by f. (*a*) $\alpha = 1$. (*b*) $\alpha = 0{\cdot}7$. (*c*) $\alpha = 0{\cdot}5$. (*d*) $\alpha = 0{\cdot}3$. (*e*) $\alpha = 0.2$.

that 'back scatter' generation of Kelvin waves predominates with increasing frequency σ. Intermediate values of α $(0{\cdot}3 < \alpha < 0{\cdot}5)$ provide a uniform transition between these extremes.

Similarly for fixed frequency σ the figures illustrate that as the correlation scale α of the irregularities decreases, back scatter becomes progressively more

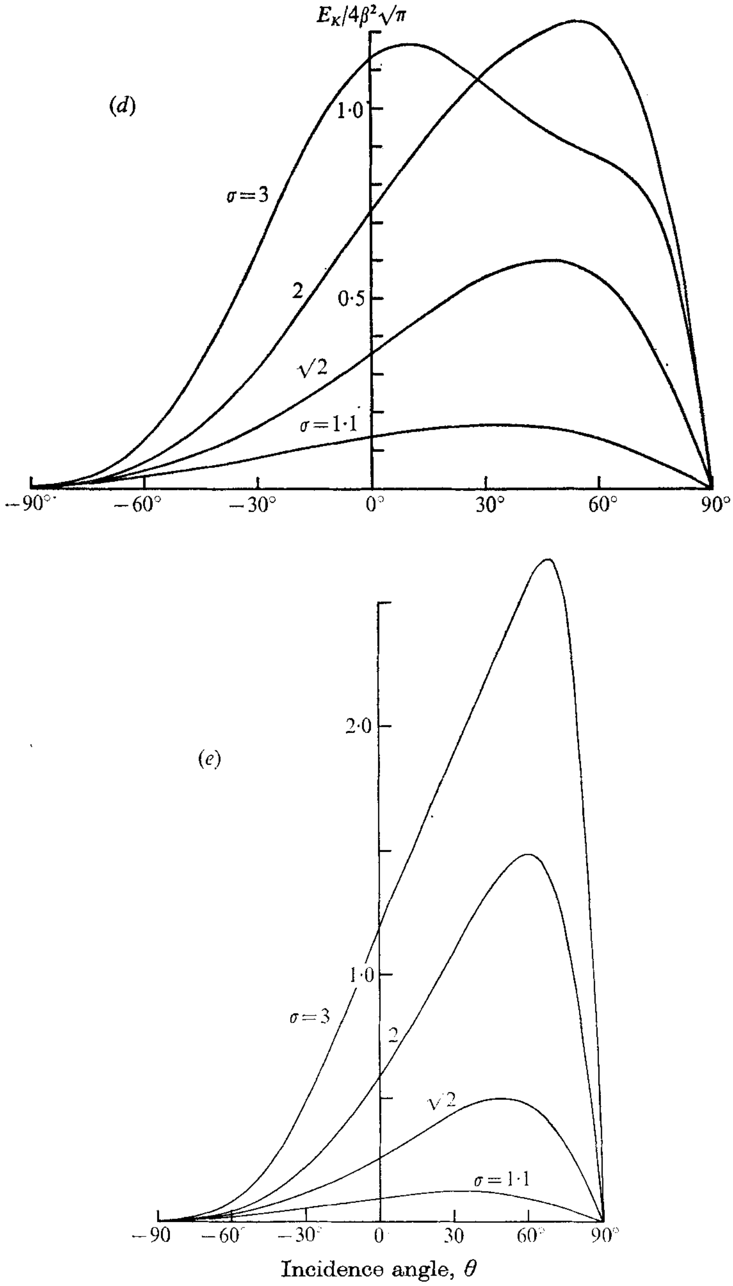

FIGURE 2 (d, e). For legend see previous page.

efficient, and also the peaks of the maxima sharper, indicating a tendency for the Kelvin wave scattering mechanism to become highly selective.

Now turn attention to the energy partition ratio Θ given by (5.12). The dependence of Θ on α shown explicitly in (5.12) implies that for small correlation scales Θ might also be quite small, most of the scattered energy therefore being directed into the Poincaré ocean wave noise. We can analyse this a little more precisely by considering the variation of Θ with the frequency σ in the two extremes of large and small α.

When the correlation length α is large compared with the incident wavelength, the spectrum function S appearing in the integral of (5.13) decays to zero very rapidly for relatively small departures of $\mu - \alpha\gamma \sin\theta$ from zero. The integral may therefore be estimated in this case by expanding about $\mu = \alpha\gamma\sin\theta$. When further $\sigma \gg 1$, we finally deduce that

$$\Theta \simeq \frac{\pi\alpha(1+\sin\theta)^2}{\cos^3\theta} S(\alpha\sigma[1+\sin\theta]) \quad (\sigma \gg 1), \tag{5.16}$$

provided that $\theta \neq \pm\frac{1}{2}\pi$. It is clear that in this limit the relative efficiency of Kelvin wave generation will tend to be rather small because of the large argument of the spectrum S.

In the long wavelength limit, however, $\sigma \to +1$ and the above approximation is no longer appropriate. In this case $\gamma \to 0$ and the range of integration in (5.13) becomes a small interval about $\lambda = 0$, from which we deduce that $I \simeq 2\alpha^2\gamma^2\sigma^2 S(0)$, and (5.12) then yields

$$\Theta \simeq \frac{\sigma}{\sigma^2-1}\frac{\pi S(\alpha\sigma)}{2S(0)}. \tag{5.17}$$

Now Θ can be very large, indicating a preferential generation of Kelvin waves.

It appears, therefore, that an incident Poincaré wave whose wavelength is short compared with the correlation scale will scatter preferentially a field of Poincaré modes. Longer wavelength incident waves, however, generally result in a strong Kelvin wave field, except possibly for very 'smooth' coasts for which $S(\alpha\sigma)$ could still be quite small.

6. Concluding remarks

Let us now summarize the main conclusions of the geophysical discussion of § 5.

The important dimensionless parameter governing the magnitude of the energy extracted from an incident Poincaré wave interacting with an irregular coastline is $\beta = (\overline{\xi^2})^{\frac{1}{2}} f/s$. The intensity of the scattering into both the Kelvin and the Poincaré modes is proportional to β^2. In situations where the present theory is likely to be applicable β rarely exceeds 0·1, so that in general the proportion of the incident energy actually lost to the scattered field, and represented analytically by E_K and E_P, tends to be rather small. Of course the relative smallness of these factors does not imply that the energy contained in these fields is necessarily insignificant. Indeed energy scattered into the Kelvin mode would be expected to remain in a fairly narrow channel along the coast, so that even quite a moderate power flux could conceivably maintain a wave system of considerable amplitude.

Analysis of a Gaussian model of the coastal spectrum function indicates that in the case of a relatively smooth coast, as measured on a scale of $(gh)^{\frac{1}{2}}/f$, Kelvin waves are preferentially generated by incident waves propagating in the same direction as the Kelvin modes ('forward scatter'). More irregular coastlines, however, for which the correlation length is much shorter, tend to extract rather more of the energy necessary to support the Kelvin wave motions from incident waves propagating in the opposite direction (strong 'back scatter'), and also exhibit a tendency to be more sharply tuned in the sense that, as the correlation scale diminishes, only waves incident in progressively narrower bands of direction of propagation actually contribute to this power flux.

Examination of the partition ratio Θ implies that Kelvin waves are scattered in preference to Poincaré waves when the frequency of the incident wave is rather small, the reverse being the case at higher frequencies. For reasons discussed above, however, this should not be taken as a measure of the relative magnitudes of the scattered wave fields. Actually the intensity of the Kelvin field, for example, is determined by the competing influences of (i) input from the incident Poincaré waves, (ii) losses due to attenuation mechanisms in the coastal boundary layer, and (iii) losses due to further scattering. An analysis of these composite effects is currently being undertaken and will appear in a future publication, but it may be noted that (iii) is expected to be important only over distances of $O(1/\beta^2)$, from which it may be argued that the attentuation mechanisms in (ii) are likely to be of more significance.

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